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Unitary random-matrix ensemble with governable level confinement

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A family of unitary α ensembles of random matrices with governable confinement potential $V(x) \sim |x|^{\alpha}$ is studied employing exact results of the theory of nonclassical orthogonal polynomials. The density of levels, two-point kernel, locally rescaled two-level cluster function, and smoothed connected correlations between the density of eigenvalues are calculated for strong ($\alpha > 1$) and border ($\alpha = 1$) level confinement. It is shown that the density of states is a smooth function for $\alpha > 1$, and has a well-pronounced peak at the band center for $\alpha \leq 1$. The case of border level confinement associated with transition point $\alpha = 1$ is reduced to the exactly solvable Pollaczek random-matrix ensemble. Unlike the density of states, all the two-point correlators remain (after proper rescaling) universal down to and including $\alpha = 1$.

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I. INTRODUCTION

Random-matrix theory [1] describes a great variety of physical systems from complex nuclei and classically chaotic systems to electron transport in disordered conductors. The latter are known to exhibit crossover from metallic phase with Wigner-Dyson level statistics to an insulating one with Poisson level statistics as disorder increases, passing through the Anderson transition where a third universal statistics should exist [2].

There are two ways to describe such a crossover within the random-matrix theory. The first one deals with random-matrix ensembles which do not hold their invariance under orthogonal, unitary, or symplectic transformations from the beginning [3] due to the presence of a symmetry-breaking term in the joint probability density function $P[\mathbf{H}]$ of the $N \times N$ elements of the random matrix $\mathbf{H}$. Such random-matrix ensembles with primordially broken symmetry demonstrate a deviation of the level statistics from the Wigner-Dyson type to the Poissonian one.

The second approach starts with invariant ensembles of random matrices that implies the logarithmic repulsion between the levels confined by a parameter-dependent potential. In this case [1] a rigorous treatment in the terms of orthogonal polynomials turns out to be successful if those are known for the chosen confinement potential. For instance, in the model based on the nonclassical q polynomials [4,5] the relevant parameter, entering the confinement potential, is associated with the strength of disorder. The strong confinement was found to be relevant to weak disorder (metallic regime), while the soft confinement potential was characteristic for strong disorder (insulating regime).

Correspondence between these two approaches has been traced in [6], where it was shown that in random-matrix ensemble with quadratic logarithmic confinement potential the spontaneous breaking of underlying symmetry of $P[\mathbf{H}]$ occurs. The symmetry breaking manifests itself in the loosing of the translational invariance of the two-level cluster function $Y_2(s, s')$ and leads to the Poisson-like level statistics.

Another family of unitary ensembles of random matrices with governable confinement potential was proposed in [7], where the so-called α-ensemble was introduced. The symmetric α-ensemble was considered in [8]. This ensemble provides an excellent basis for rigorous treatment [8] and enables us to explore how soft the confinement potential must be to cause deviations from the Wigner-Dyson statistics.

Let us consider a physical system with broken time-reversal symmetry described by an $N \times N$ random matrix $\mathbf{H}$ with eigenvalues $\{x_n\}, n = 1, ..., N$. The joint probability density function $P(\{x\})$ can be written in the form [1]

$$P(\{x\}) = Z^{-1} \exp\left[-\beta\left(\sum_i V(x_i) - \sum_{i<j} \ln\left|x_i - x_j\right|\right)\right], \quad \beta = 2, \tag{1}$$

which implies a pairwise logarithmic repulsion between the levels confined by the potential $V(x)$; Z is a partition function.

Symmetric α-ensemble is characterized by parameter-dependent potential

$$V_\alpha(x) = \frac{1}{2}|x|^{\alpha} \tag{2}$$

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supported on the whole real axis $x \in]-\infty, +\infty[$.

The Monte Carlo simulations [8], based on the mapping of the initial random-matrix problem onto the problem of interacting particles in confined 1D plasma, showed that in such α ensemble the deviations from the Wigner-Dyson statistics may also occur, and they take place near the center of the spectrum for $0 < \alpha < 1$. The value $\alpha = 1$ was associated with the sharp transition point corresponding to the crossover from strong to soft level confinement. Just at this point, $\alpha = 1$, the mean-field approximation widely used in the random-matrix theory fails, giving rise to the singularity of the level density near the spectrum origin.

In this paper we present the analytical treatment of α-ensemble. In Sec. II α ensemble with strong confinement potential ($\alpha > 1$) is studied employing exact results of the theory of nonclassical orthogonal polynomials. In Sec. III we describe the general properties of symmetric α ensemble revealing the fact that $\alpha = 1$ is a special point associated with peak formation in the density of states. This special case of *border* level confinement is investigated analytically in Sec. IV in the framework of the theory of orthogonal polynomials, exploiting the properties of the symmetric Pollaczek polynomials. In Secs. III and IV we find the density $\nu_N(x)$ of levels, two-point kernel $K_N(x,y)$, locally rescaled two-level cluster function $Y_2(s,s')$, and smoothed connected correlations $[\nu_N(x,y)]_{con}$ between the density of eigenvalues. It will be shown that for strong and border level confinement the rescaled local two-level cluster functions as well as the smoothed connected correlations follow the universal forms which are typical of random-matrix ensembles with the Wigner-Dyson level statistics [9–11]. Section V contains conclusions.

II. SYMMETRIC α ENSEMBLE: CASE OF STRONG LEVEL CONFINEMENT ($\alpha > 1$)

The rigorous treatment of α ensemble with strong level confinement has been made possible by the recent development of the theory of nonclassical orthogonal polynomials. The relevant polynomials $P_n^{(\alpha)}(x)$ are known as those orthogonal with respect to the Freud weights [12,13].

All the n-point correlation functions can be expressed through the two-point kernel [1]

$$K_N^{(\alpha)}(x,y) = e^{-V_\alpha(x)-V_\alpha(y)}\frac{k_{N-1}}{k_N} \times \frac{P_{N-1}^{(\alpha)}(x)\,P_N^{(\alpha)}(y) - P_{N-1}^{(\alpha)}(y)\,P_N^{(\alpha)}(x)}{y-x}, \tag{3}$$

where k_N is a leading coefficient of the orthonormal polynomial $P_N^{(\alpha)}(x)$. The pointwise asymptotics of $P_N^{(\alpha)}(x)$ for large N have the form [14]

$$P_N^{(\alpha)}(x) = \left(\frac{2}{\pi D(N,\alpha)}\right)^{1/2} \frac{\exp\left(V_\alpha(x)\right)}{\left\{1-[x/D(N,\alpha)]^2\right\}^{1/4}} \times \cos\left(\Phi_N^{(\alpha)}(x)\right), \tag{4}$$

with

$$\Phi_N^{(\alpha)}(x) = \pi N \int_{x/D(N,\alpha)}^{1} \omega_\alpha(z)\,dz + \frac{1}{2}\arccos\left(\frac{x}{D(N,\alpha)}\right) - \frac{\pi}{4}. \tag{5}$$

Here

$$\omega_\alpha(z) = \frac{\alpha}{\pi}|z|^{\alpha-1}\int_{|z|}^{1}\frac{d\eta}{\eta^\alpha\sqrt{1-\eta^2}} \tag{6}$$

is the Nevai-Ullman density,

$$D(N,\alpha) = N^{1/\alpha}D(\alpha)\,,\ D(\alpha) = \left(\frac{2^{\alpha-1}\Gamma^2\left(\frac{\alpha}{2}\right)}{\Gamma(\alpha)}\right)^{1/\alpha}, \tag{7}$$

and

$$k_{N-1}/k_N = \frac{1}{2}D(N,\alpha)\,. \tag{8}$$

The straightforward calculations based on Eqs. (3)–(8) lead to the following expression:

$$K_N^{(\alpha)}(x,y) = \frac{1}{\pi(y-x)}\left\{\left[1-\left(\frac{x}{D(N,\alpha)}\right)^2\right]\left[1-\left(\frac{y}{D(N,\alpha)}\right)^2\right]\right\}^{-1/4} \times\left[\cos\left(\Phi_{N-1}^{(\alpha)}(x)\right)\cos\left(\Phi_N^{(\alpha)}(y)\right) - \cos\left(\Phi_N^{(\alpha)}(x)\right)\cos\left(\Phi_{N-1}^{(\alpha)}(y)\right)\right]. \tag{9}$$

Despite the fact that this function is rather complicated, it can be rewritten in the universal form if one supposes that $|x-y|$ is much smaller than the scale ς of characteristic changes of the mean level density, $\varsigma \sim \nu_N^{(\alpha)}\left(d\nu_N^{(\alpha)}/dx\right)^{-1} \sim D(N,\alpha)$. Assuming also that both x and y stay away from the band edge $D(N,\alpha)$ of the spectrum and making use of the asymptotic identity

$\Phi_{N-1}^{(\alpha)}(x) = \Phi_N^{(\alpha)}(x) - \arccos(x/D(N,\alpha))$, we obtain in the leading order in $1/N$

$$K_N^{(\alpha)}(x,y) = \frac{\sin\left(\pi\bar{\nu}_N^{(\alpha)}(x-y)\right)}{\pi(x-y)}, \tag{10}$$

where

$$\bar{\nu}_N^{(\alpha)} = \frac{N}{D(N,\alpha)}\omega_\alpha\left(\frac{x+y}{2D(N,\alpha)}\right) \tag{11}$$

is the local density of levels. Note that Eq. (10) is valid on the scale $|x-y|$ which is much larger than the mean level spacing $\left(\bar{\nu}_N^{(\alpha)}\right)^{-1} \sim N^{1/\alpha-1}$.

Correspondingly, in the large-N limit the two-level cluster function

$$Y_2(s,s') = \left(\frac{K_N^{(\alpha)2}(x,y)}{\nu_N^{(\alpha)}(x)\,\nu_N^{(\alpha)}(y)}\right)_{\substack{x=x(s)\\ y=y(s')}}, \tag{12}$$

being rewritten in the terms of eigenvalues measured in the local level spacing $s = x\bar{\nu}_N^{(\alpha)}$ and $s' = y\bar{\nu}_N^{(\alpha)}$, locally follows the universal form

$$Y_2(s,s') = \frac{\sin^2[\pi(s-s')]}{[\pi(s-s')]^2} \tag{13}$$

irrespective to the value $\alpha > 1$.

As is well known [1], the level-spacing distribution function $P(\Delta)$ can be expressed through the eigenvalues $\{\lambda(\Delta)\}$ of the Fredholm integral equation where $\sqrt{Y_2} = \sin[\pi(s-s')]/\pi(s-s')$ stands for the kernel

$$\int_{-\Delta/2}^{\Delta/2} ds'\sqrt{Y_2(s,s')}f(s') = \lambda(\Delta)f(s), \tag{14}$$

$$P(\Delta) = \frac{d^2}{d\Delta^2}\prod_j[1-\lambda_j(\Delta)]. \tag{15}$$

Since $Y_2(s,s')$ for α ensemble coincides with that for Gaussian ensemble, it inevitably leads to universal Wigner-Dyson statistics.

The connected correlations between the density of eigenvalues for $x \neq y$ are given by [9]

$$\left[\nu_N^{(\alpha)}(x,y)\right]_{con} = -K_N^{(\alpha)2}(x,y) \tag{16}$$

and are known to oscillate rapidly on the scale of the bandwidth $D(N,\alpha)$. The smoothed correlation function which is useful for the calculation of integral characteristic of spectrum can easily be determined. Bearing in mind Eq. (9) we obtain after averaging over rapid oscillations

$$\left[\nu_N^{(\alpha)}(x,y)\right]_{con} = -\frac{1}{2\pi^2(x-y)^2} \times \frac{D(N,\alpha)^2 - xy}{\sqrt{\left[D(N,\alpha)^2 - x^2\right]\left[D(N,\alpha)^2 - y^2\right]}}, \quad x \neq y. \tag{17}$$

Equation (17) proves that smoothed correlations in α ensemble with strong level confinement follow universal form [9,10].

Note that along with the proof of universality of eigenvalue correlations we have obtained an *exact* formula for the density of levels in α ensemble. Thus, Eqs. (6) and (11) yield the following expression for $\alpha > 1$:

$$\nu_N^{(\alpha)}(x) = \frac{\alpha\Gamma(\alpha)}{2^{\alpha-1}\pi\Gamma^2\left(\frac{\alpha}{2}\right)}|x|^{\alpha-1}\int_{|z|}^{1}\frac{d\eta}{\eta^\alpha\sqrt{1-\eta^2}}, \tag{18}$$

that can be rewritten in two equivalent ways by means of hypergeometric functions,

$$\nu_N^{(\alpha)}(x) = \frac{\alpha}{\pi}\frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\alpha}{2}\right)}|x|^{\alpha-1}\sqrt{1-z^2} \times {}_2\mathbf{F}_1\left(\frac{1}{2},\frac{1+\alpha}{2};\frac{3}{2};1-z^2\right), \tag{19}$$

or

$$\nu_N^{(\alpha)}(x) = \frac{\alpha N^{1-1/\alpha}}{\pi D(\alpha)}\sqrt{1-z^2}\,{}_2\mathbf{F}_1\left(1,1-\frac{\alpha}{2};\frac{3}{2};1-z^2\right), \tag{20}$$

where $z = x/D(N,\alpha)$.

In the particular case $\alpha = 2$ Eq. (20) recovers the famous Wigner's semicircle, $\nu_N^{(2)}(x) = \pi^{-1}\sqrt{2N-x^2}$.

It is worth pointing out that Eqs. (19) and (20) obtained within the framework of the theory of orthogonal polynomials exactly coincide with the density of states calculated for symmetric α ensemble in the mean-field approximation [8]. This circumstance justifies the validity of the mean-field approach for $\alpha > 1$. For pure linear confinement potential ($\alpha = 1$) the asymptotic formula Eq. (4) fails, signaling that point $\alpha = 1$ is a special one.

III. GENERAL PROPERTIES OF SYMMETRIC α ENSEMBLE

The special character of the point $\alpha = 1$ may be understood appealing to the recent mathematical literature on the theory of orthogonal polynomials with respect to the Freud weights [15]. Noting that the density of levels ν_N for random-matrix ensemble with confinement potential $V(x)$ is related to an inverse Christoffel function $\lambda_N^{-1}(x) = \sum_{i=0}^{N-1} P_i^2(x)$ for polynomials $P_i(x)$ orthogonal with respect to the weight $w(x) = \exp(-2V(x))$ as

$$\nu_N(x) = e^{-2V(x)}\lambda_N^{-1}(x), \tag{21}$$

we conclude that the crucial changes occur in the N-dependence of the level density at the origin [15]

$$\nu_{orig}(N,\alpha) \sim \begin{cases} N^0, & 0<\alpha<1, \\ \ln N, & \alpha=1, \\ N^{1-1/\alpha}, & \alpha>1. \end{cases} \tag{22}$$

Thus, at the point $\alpha = 1$ the functional dependence of the density of states on the number of levels differs from that both for $0 < \alpha < 1$ and $\alpha > 1$.

The ratio

$$\xi(\alpha) = \lim_{N\to\infty} \frac{\nu_{bg}(N,\alpha)}{\nu_{orig}(N,\alpha)}, \tag{23}$$

with ν_{bg} being the background density of levels, demonstrates all the more dramatic behavior. The ν_{bg} can be estimated as $N/2D(N,\alpha)$, where $D(N,\alpha)$ is the band edge for symmetric α ensemble, and N is a normalization of the level density. The estimates of Christoffel functions [15] allow us to relate $D(N,\alpha)$ to the maximal zero x_{1N} of the corresponding Nth orthogonal polynomials, $x_{1N} \sim N^{1/\alpha}$, so that $\nu_{bg} \sim N^{1-1/\alpha}$. Such a definition of ν_{bg} is relevant not only in the case of strong level confinement, $\alpha > 1$, when density of levels in the spectrum bulk is smooth, but also in the case of weak level confinement although the bulk level density is no longer constant. In the latter case the density of levels in the spectrum bulk is very well approximated by $1/|x|^{1-\alpha}$ [8]. Since in the bulk of the spectrum $|x| = \epsilon D(N,\alpha)$ with $0 < \epsilon < 1$ we obtain $\nu_{bg}(N,\alpha) \sim \epsilon^{\alpha-1} D(N,\alpha)^{\alpha-1} \sim N^{1-1/\alpha}$. This estimate is fully consistent with the definition $\nu_{bg} = N/2D(N,\alpha)$ given above. The circumstance that $\nu_{bg}(N,\alpha) \to 0$ as $N \to \infty$ reflects only the fact that the case of weak level confinement is characterized by the low level density and wide spectrum support; despite that ν_{bg} tends to zero in the large-N limit, the normalization condition $N = 2\nu_{bg}(N,\alpha)\,D(N,\alpha)$ is fulfilled thanks to the fact that spectrum edge $D(N,\alpha)$ goes to infinity as $N^{1/\alpha}$.

Now we immediately obtain from Eq. (22) that $\xi(\alpha) = 0$ for $0 < \alpha \le 1$. The case $\alpha > 1$ can be treated with the aid of Eq. (20) which yields the density of levels at the origin ($x = 0$),

$$\nu_{orig}(N,\alpha) = \frac{\alpha N^{1-1/\alpha}}{\pi(\alpha-1)\,D(\alpha)}, \tag{24}$$

so that $\xi(\alpha) = \frac{\pi}{2}(1 - 1/\alpha)$ for $\alpha > 1$. Finally, we obtain

$$\xi(\alpha) = \begin{cases} 0, & 0<\alpha\le 1, \\ \frac{\pi}{2}\left(1-\frac{1}{\alpha}\right), & \alpha>1. \end{cases} \tag{25}$$

The function $\xi(\alpha)$ is plotted in Fig. 1.

Equation (25) implies that the transition point $\alpha = 1$ corresponds to the formation of a sharp peak at the spectrum origin which also holds for $0 < \alpha < 1$ (soft confinement), but is absent in the case of strong confinement potential, $\alpha > 1$.

IV. BORDER LEVEL CONFINEMENT ($\alpha = 1$)

A. Pollaczek random-matrix ensemble (PRME)

The formalism developed in Sec. II cannot be directly applied to the case $\alpha = 1$, since the relevant pointwise asymptotic formula Eq. (4) fails. Nevertheless, this difficulty can be avoided by choosing $V(x)$ in the form

$$V^{(\lambda)}(x) = -\frac{1}{2}\ln w^{(\lambda)}(x) = \frac{1}{2}\sum_{k=0}^{\infty} \ln\left(1 + \frac{x^2}{(k+\lambda)^2}\right) + V^{(\lambda)}(0), \tag{26}$$

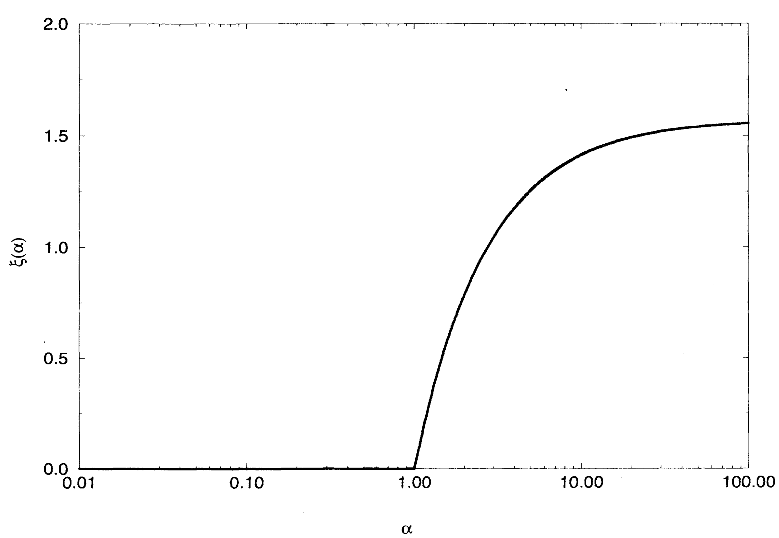

FIG. 1. Function $\xi(\alpha)$ demonstrating the sharp peak formation in the density of states at the origin of the spectrum when $\alpha \le 1$.

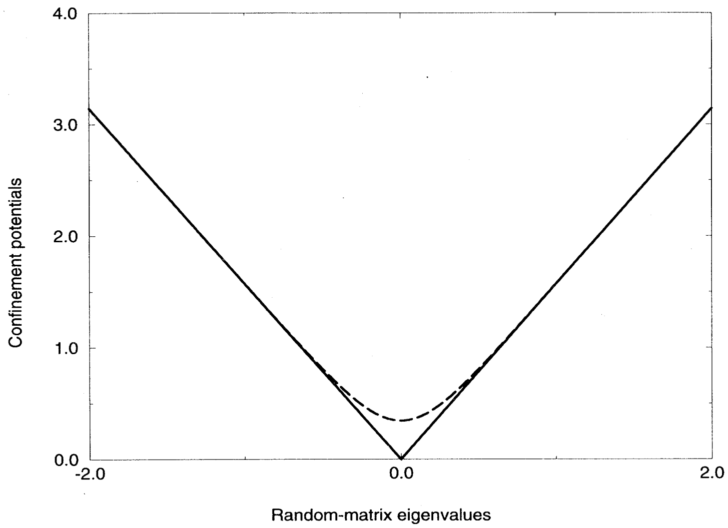

FIG. 2. Confinement potential $V^{(1/2)}(x) + V_\infty^{(1/2)}$ for Pollaczek random-matrix ensemble with $\lambda = 1/2$ (dashed line) and linear potential $V_L(x) = \pi|x|/2$ (solid line). The only small discrepancy takes place in the narrow region $|x| < 1$.

where $w^{(\lambda)}(x)$ is the weight function for Pollaczek polynomials. Their basic properties are collected in Appendix A.

The behavior of confinement potential $V^{(\lambda)}(x)$ can easily be obtained from Eqs. (26) and (A2). In the vicinity of the origin $x = 0$, $|x| \ll \lambda$, expansion of Eq. (26) yields the quadratic potential

$$V^{(\lambda)}(x) \approx V^{(\lambda)}(0) + \frac{1}{2}x^2\Psi^{(1)}(\lambda), \tag{27}$$

where $\Psi^{(1)}(\lambda) = \sum_{k=0}^{\infty}(k+\lambda)^{-2}$ is a trigamma function. The long-range behavior of $V^{(\lambda)}(x)$ follows from the asymptotic representation of $|\Gamma(\lambda + ix)|$ when $|x| \to \infty$ [see Eq. (A2)], and turns out to be

$$V^{(\lambda)}(x) \approx \frac{\pi}{2}|x| - \left(\lambda - \frac{1}{2}\right)\ln|x| - V_\infty^{(\lambda)} \tag{28}$$

with $V_\infty^{(\lambda)} = \lambda \ln 2$.

Thus, the confinement potential in PRME exhibits such a long-range behavior which involves, in particular, the precisely linear growth at large $|x|$ if one puts $\lambda = 1/2$. The calculations show (see Fig. 2) that only a small discrepancy between linear confinement potential $V_L(x) = \pi|x|/2$ and $V^{(1/2)}(x)$ takes place in the region $|x| < 1$, which is negligible as compared with the band edge $D(N,1) = N$. This circumstance allows us to correlate PRME with the transition point $\alpha = 1$ of the α ensemble mentioned in the Introduction. The advantage of the proposed unitary ensemble of random matrices is that it can be treated *exactly* in the terms of orthogonal Pollaczek polynomials. (For the sake of convenience, the derivation of asymptotic properties of these polynomials is entered in Appendix B.)

B. Density of states

The density of states can be calculated making use of the asymptotics of Pollaczek polynomials found in Appendix B. Namely, the density of states for eigenvalues of PRME reads [1]

$$\nu_N^{(\lambda)}(x) = e^{-2V^{(\lambda)}(x)}\frac{k_{N-1}}{k_N h_{N-1}^{(\lambda)}}\left(P_{N-1}^{(\lambda)}(x)\frac{d}{dx}P_N^{(\lambda)}(x) - P_N^{(\lambda)}(x)\frac{d}{dx}P_{N-1}^{(\lambda)}(x)\right). \tag{29}$$

Bearing in mind Eqs. (A2), (A4), (A5), and different asymptotics for $P_n^{(\lambda)}(x)$ near the origin of the spectrum [Eq. (B14)] and in its bulk [Eq. (B9)], we easily obtain for finite λ in the limit $N \gg 1$

$$\nu_N^{(\lambda)}(x) = \frac{1}{\pi}\left[\ln(2N) - \mathrm{Re}\Psi(\lambda + ix)\right], \quad |x| \ll \sqrt{2N}, \tag{30}$$

$$\nu_N^{(\lambda)}(x) = \frac{1}{2\pi}\ln\left(\frac{1+\sqrt{1-(x/N)^2}}{1-\sqrt{1-(x/N)^2}}\right), \quad 1 \ll |x| < N. \tag{31}$$

Here $\Psi(z) = (d/dz)\ln\Gamma(z)$ is a digamma function.

Equations (30) and (31) lead to the conclusion that the density of states for PRME does not depend on the parameter λ in the bulk of the spectrum, whereas at the origin this λ dependence holds. Such a behavior of the density of states is not a surprise and is known for the generalized Gaussian and Laguerre ensembles [16].

As $N \to \infty$ the density of states at the origin tends asymptotically to the value $\nu_N^{(\lambda)}(0) = [\ln(2N) - \Psi(\lambda)]/\pi$. We note that this result is in agreement with $\nu_{orig}(N,1)$, obtained for strictly linear potential $V_L(x)$, see Eq. (22). The nonasymptotic formula for $\nu_N^{(\lambda)}(0)$ that is

$$\nu_N^{(\lambda)}(0) = e^{-2V^{(\lambda)}(0)} \sum_{j=0}^{N-1} \frac{1}{h_j^{(\lambda)}} \left| P_j^{(\lambda)}(0) \right|^2 \qquad (32)$$

can also be obtained. From recurrence Eq. (A1) it follows that

$$P_{2j+1}^{(\lambda)}(0) = 0,\ P_{2j}^{(\lambda)}(0) = (-1)^j \frac{\Gamma(j+\lambda)}{\Gamma(\lambda)\Gamma(j+1)}. \qquad (33)$$

Therefore we arrive at the relationship

$$\nu_N^{(\lambda)}(0) = \frac{1}{\pi} \sum_{j=0}^{\left[\frac{N-1}{2}\right]} \frac{\Gamma\left(j+\frac{1}{2}\right)\Gamma(j+\lambda)}{\Gamma(j+1)\Gamma\left(j+\lambda+\frac{1}{2}\right)}. \qquad (34)$$

Here Eqs. (A2) and (A4) were used, and $[m]$ stands for integer part of m.

Note that the density of states given by Eq. (31), being extended onto the whole interval $x \in [-N, N]$ of the eigenvalues of PRME, has logarithmic singularity at the origin, but it still remains integrable and obeys normalization condition

$$\int_{-N}^{+N} dx \nu_N^{(\lambda)}(x) = N. \qquad (35)$$

Moreover, Eq. (31) exactly coincides with the density of states for confinement potential $V_L(x) = \pi|x|/2$ which can be found within the mean-field approximation [8]. This circumstance is a strong evidence that the mean-field approach, which was proved to be valid for $V(x) \sim |x|^\alpha$ with $\alpha > 1$ (see Sec. II), is justified for weaker *symmetric* potentials up to linearlike (except for the region close to the origin of the spectrum).

The density of states is presented in Fig. 3, where an excellent coincidence is observed between analytical asymptotic expressions Eqs. (30), (31), and the density of states, calculated from the hypergeometric representation of Pollaczek polynomials [see Eqs. (A2) and (A6)]:

$$\nu_N^{(\lambda)}(x) = \frac{2^{2\lambda-1}}{\pi} |\Gamma(\lambda+ix)|^2 \times \sum_{n=0}^{N-1} \left| \frac{(2\lambda)_n}{n!}\, {}_2\mathbf{F}_1(-n, \lambda+ix; 2\lambda; 2) \right|^2. \qquad (36)$$

We also computed the density of states for pure linear confinement potential $V_L(x)$ using the well-known (in the theory of orthogonal polynomials) matrix representation for Christoffel function [17],

$$\widetilde{\nu}_N(x) = -e^{-2V_L(x)} \det \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{N-1} & 1 \\ \mu_1 & \mu_2 & \dots & \mu_N & x \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \mu_{N-1} & \mu_N & \dots & \mu_{2N-1} & x^{N-1} \\ 1 & x & \dots & x^{N-1} & 0 \end{pmatrix} \times \left[\det \begin{pmatrix} \mu_0 & \mu_1 & \dots & \mu_{N-1} \\ \mu_1 & \mu_2 & \dots & \mu_N \\ \cdot & \cdot & \cdot & \cdot \\ \mu_{N-1} & \mu_N & \dots & \mu_{2N-1} \end{pmatrix} \right]^{-1}, \qquad (37)$$

where

$$\mu_k = \int_{-\infty}^{+\infty} x^k \exp(-2V_L(x))dx = \frac{\left[1+(-1)^k\right]}{\pi^{k+1}} \Gamma(k+1). \qquad (38)$$

The results plotted in Fig. 4 constitute additional justifications of the use of confinement potential $V^{(1/2)}(x)$ instead of the initial linear one, $V_L(x)$.

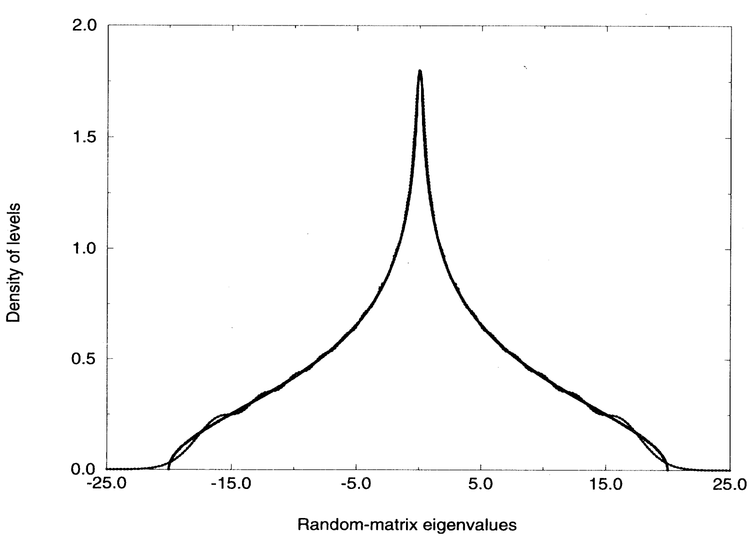

FIG. 3. Density of levels for PRME. Solid line: asymptotic formulas Eqs. (30) and (31). Dotted line: hypergeometric representation Eq. (36). Parameters: $\lambda = 1/2$, $N = 20$.

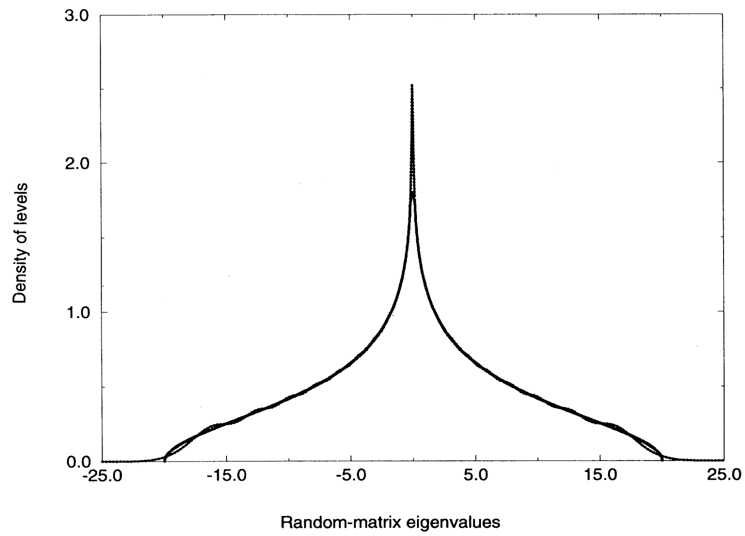

FIG. 4. Density of levels for confinement potential $V_L(x) = \pi|x|/2$ calculated from Eqs. (37) and (38) (dotted line) and for Pollaczek random-matrix ensemble with $\lambda = 1/2$ [Eqs. (30) and (31), solid line]. $N = 20$.

C. Two-point correlators

The two-point kernel can be determined as [1]

$$K_N^{(\lambda)}(x,y) = e^{-V^{(\lambda)}(x)-V^{(\lambda)}(y)} \frac{k_{N-1}}{k_N h_{N-1}^{(\lambda)}} \frac{P_{N-1}^{(\lambda)}(x)\, P_N^{(\lambda)}(y) - P_N^{(\lambda)}(x)\, P_{N-1}^{(\lambda)}(y)}{y-x}. \tag{39}$$

Here, again, different asymptotics must be used for the bulk of the spectrum and for its origin.

In the bulk of the spectrum, $1 \ll x < N$, and $1 \ll y < N$, Eq. (B9) gives

$$\begin{aligned} K_N^{(\lambda)}(x,y) = \frac{1}{\pi(y-x)} \Bigg(&\left(\frac{1-(y/N)^2}{1-(x/N)^2}\right)^{1/4} \sin\left(\Phi_N^{(\lambda)}(x)\right) \cos\left(\Phi_N^{(\lambda)}(y)\right) \\ &- \left(\frac{1-(x/N)^2}{1-(y/N)^2}\right)^{1/4} \sin\left(\Phi_N^{(\lambda)}(y)\right) \cos\left(\Phi_N^{(\lambda)}(x)\right) - \frac{y-x}{N} \\ &\times \left\{ \left[1-\left(\frac{x}{N}\right)^2\right]\left[1-\left(\frac{y}{N}\right)^2\right] \right\}^{-1/4} \sin\left(\Phi_N^{(\lambda)}(x)\right) \sin\left(\Phi_N^{(\lambda)}(y)\right) \Bigg), \end{aligned} \tag{40}$$

where

$$\Phi_N^{(\lambda)}(x) = \frac{\pi}{4} + (N+\lambda)\arccos\left(\frac{x}{N}\right) - \pi x \nu_N^{(\lambda)}(x). \tag{41}$$

It can be seen that despite the fact that this function has a rather complicated form, the *local* properties of the two-point kernel remain universal. Really, for $|x-y|$ that is much smaller than the scale ς of characteristic changes of the mean level density, $\varsigma \sim \nu_N^{(\lambda)} \left(d\nu_N^{(\lambda)}/dx\right)^{-1} \sim N$, and both x and y at finite distance from the edge of the spectrum, the third term can be neglected. Then, one obtains

$$K_N^{(\lambda)}(x,y) = \frac{1}{\pi(y-x)} \sin\left[\Phi_N^{(\lambda)}(x) - \Phi_N^{(\lambda)}(y)\right]. \tag{42}$$

Making use of the different representation for $\Phi_N^{(\lambda)}$,

$$\Phi_N^{(\lambda)}(x) = \frac{\pi}{4}(1+2N) + \lambda \arccos\left(\frac{x}{N}\right) - \pi \int_0^x \nu_N^{(\lambda)}(z)\, dz, \tag{43}$$

with $\nu_N^{(\lambda)}$ defined by Eq. (31), we easily obtain the universal form of the two-point kernel in the limit $N \to \infty$:

$$K_N^{(\lambda)}(x,y) = \frac{1}{\pi(x-y)} \sin\left(\pi \overline{\nu}_N^{(\lambda)}(x-y)\right), \tag{44}$$

where $\overline{\nu}_N^{(\lambda)} = \nu_N^{(\lambda)}[(x+y)/2]$ is a local mean level density. Equation (44) proves the local universality of the two-point kernel and is valid on the scale $|x-y|$ which is much larger than the mean level-spacing $\left(\overline{\nu}_N^{(\lambda)}\right)^{-1} \sim 1$.

In the vicinity of the spectrum center, $|x| \ll \sqrt{2N}$, and $|y| \ll \sqrt{2N}$, the asymptotic formula Eq. (B14) and Eq. (39) yield

$$K_N^{(\lambda)}(x,y) = \frac{1}{\pi(y-x)} \sin\left[\phi_N^{(\lambda)}(y) - \phi_N^{(\lambda)}(x)\right], \quad (45)$$

$$\phi_N^{(\lambda)}(x) = x\ln(2N) - \arg\Gamma(\lambda + ix) = \pi \int_0^x \nu_N^{(\lambda)}(z)\, dz, \quad (46)$$

where $\nu_N^{(\lambda)}$ is determined by Eq. (30). Since near the origin the scale ς of characteristic changes of the mean level density is of the order λ, Eq. (45) can also be rewritten in the universal form Eq. (44) but with density of states $\nu_N^{(\lambda)}$ corresponding to the center of the spectrum. Here, again, universal form of the two-point kernel is valid on the scale $|x-y|$ which is much larger than the mean level-spacing $\left(\overline{\nu}_N^{(\lambda)}\right)^{-1} \sim 1/\ln N$.

Thus, we arrive at the conclusion that the two-level cluster function for PRME,

$$Y_2(s,s') = \left(\frac{K_N^{(\lambda)2}(x,y)}{\nu_N^{(\lambda)}(x)\,\nu_N^{(\lambda)}(y)}\right)_{\substack{x=x(s)\\ y=y(s')}}, \quad (47)$$

being rewritten in the terms of the eigenvalues measured in the local level spacing $s = x\overline{\nu}_N^{(\lambda)}$ and $s' = y\overline{\nu}_N^{(\lambda)}$, locally takes the universal form Eq. (13) for any finite λ.

The connected correlations between the density of eigenvalues are determined as

$$\left[\nu_N^{(\lambda)}(x,y)\right]_{con} = -K_N^{(\lambda)2}(x,y) \quad (48)$$

if $x \neq y$ and oscillate rapidly on the scale of the bandwidth $D(N,1) = N$. The smoothed correlation function can easily be determined. Bearing in mind Eq. (40) we obtain in the bulk of the spectrum after averaging over rapid oscillations

$$\left[\nu_N^{(\lambda)}(x,y)\right]_{con} = -\frac{1}{2\pi^2(x-y)^2} \times \frac{D(N,1)^2 - xy}{\sqrt{\left[D(N,1)^2 - x^2\right]\left[D(N,1)^2 - y^2\right]}}, \quad x \neq y. \quad (49)$$

The smoothed correlations near the spectrum origin can be calculated by means of Eqs. (45) and (46), and turn out to be

$$\left[\nu_N^{(\lambda)}(x,y)\right]_{con} = -\frac{1}{2\pi^2(x-y)^2}, \; x \neq y. \quad (50)$$

Equation (50) is the limited case of Eq. (49) when both $|x|$ and $|y|$ are much smaller than the band edge $D(N,1)$.

Thus, Eqs. (49) and (50) prove that smoothed correlations in PRME follow universal form.

V. CONCLUSION

We have considered the general properties of symmetric α ensemble with confinement potential $V(x) \sim |x|^\alpha$ and established that the phenomenon of the peak formation in the density of states takes place. Namely, we have demonstrated that this sharp peak occurs for $0 < \alpha \leq 1$ and is absent in the opposite case $\alpha > 1$. In this sense the point $\alpha = 1$ is a transition point associated with border level confinement. It has been shown that this transition point may be explored by means of slight changes in strictly linear confinement potential $V_L(x)$ near the spectrum origin leading to the potential $V^{(1/2)}(x) = -\frac{1}{2}\ln w^{(1/2)}(x) = \frac{1}{2}\ln\cosh(\pi x)$ which is a special case of more general confinement potential Eq. (26) connected to the symmetric Pollaczek polynomials.

We have calculated the density of states, two-point kernel, two-level cluster function, and smoothed correlations of the density of eigenvalues in the large-N limit for α ensemble with strong and border level confinement. It has been shown that both a properly rescaled two-level cluster function and smoothed correlations of level density take the universal forms which are typical of random-matrix ensembles with the Wigner-Dyson level statistics.

We have also demonstrated that the mean-field approximation is valid for calculation of the level density in symmetric α ensemble with strong level confinement, $\alpha > 1$. In the case of border level confinement, $\alpha = 1$, the mean-field approach is proved to be justified in the bulk of spectrum, failing near its origin.

We would like to stress that treatment presented in this paper is *rigorous* and does not appeal to commonly used conjectures and approximate methods.

Note added in proof: Recently, we were informed about the paper by L. A. Pastur, Lett. Math. Phys. **25**, 259 (1992) where Eqs. (10) and (18) have been obtained.

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APPENDIX A: DEFINITIONS AND BASIC PROPERTIES OF SYMMETRIC POLLACZEK POLYNOMIALS

Symmetric Pollaczek polynomials $P_n^{(\lambda)}(x)$ are determined by the recurrence equation [18,19]

$$nP_n^{(\lambda)}(x) - 2xP_{n-1}^{(\lambda)}(x) + (n-2+2\lambda)P_{n-2}^{(\lambda)}(x) = 0, \quad n = 1,2,3,\ldots \quad (A1)$$

with $P_{-1}^{(\lambda)}(x)=0, P_0^{(\lambda)}(x)=1$, and $\lambda>0$. These polynomials are orthogonal in the interval $-\infty<x<\infty$ with the weight function

$$w^{(\lambda)}(x)=\frac{2^{2\lambda-1}}{\pi}\left|\Gamma(\lambda+ix)\right|^2 =\frac{2^{2\lambda-1}\left|\Gamma(\lambda)\right|^2}{\pi}\prod_{k=0}^{\infty}\left(1+\frac{x^2}{(k+\lambda)^2}\right)^{-1}, \quad \text{(A2)}$$

so that

$$\int_{-\infty}^{+\infty}dxP_n^{(\lambda)}(x)P_m^{(\lambda)}(x)w^{(\lambda)}(x)=\delta_{nm}h_n^{(\lambda)}, \quad \text{(A3)}$$

$$h_n^{(\lambda)}=\frac{\Gamma(n+2\lambda)}{\Gamma(n+1)}. \quad \text{(A4)}$$

From recurrence equation the leading coefficient of $P_n^{(\lambda)}$ can be found:

$$k_n=\frac{2^n}{\Gamma(n+1)}. \quad \text{(A5)}$$

The following hypergeometric representation holds for Pollaczek polynomials:

$$P_n^{(\lambda)}(x)=\frac{(2\lambda)_n}{n!}\,{}_2\mathbf{F}_1(-n,\lambda+ix;2\lambda;2)\exp(i\pi n/2). \quad \text{(A6)}$$

Lastly, we present the generating function that reads

$$F(x,w)=\sum_{k=0}^{\infty}P_k^{(\lambda)}(x)w^k=\left(\frac{1-iw}{1+iw}\right)^{ix}\frac{1}{(1+w^2)^{\lambda}}, \quad |w|<1. \quad \text{(A7)}$$

APPENDIX B: ASYMPTOTIC FORMULAS FOR POLLACZEK POLYNOMIALS

1. Formula of Plancherel-Rotach type

We start with Eq. (A7) which being reversed yields

$$P_n^{(\lambda)}(x)=\frac{1}{2\pi i}\oint_{\gamma_0}\frac{dw}{w^{n+1}}F(x,w). \quad \text{(B1)}$$

Here the integration is extended over a contour γ_0 that encloses the origin $w=0$ and does not intersect the branch cuts $[i,+i\infty[$ and $]-i\infty,-i]$ associated with the singularities of generating function. Choosing $x=(n+1)\cos\theta$ with $\epsilon\le\theta\le\pi-\epsilon$, where ϵ is a fixed positive number smaller than $\pi/2$, we rewrite Eq. (B1) as

$$P_n^{(\lambda)}(x)=\frac{1}{2\pi i}\oint_{\gamma_0}\frac{dw}{(1+w^2)^{\lambda}}\times\exp\left\{(n+1)\left[i\cos\theta\ln\left(\frac{1-iw}{1+iw}\right)-\ln w\right]\right\}. \quad \text{(B2)}$$

For the calculation of this contour integral in the limit $n\gg1$ the method of steepest descent [17,20] can be applied. The saddle-point condition is

$$S'(w)=\frac{\partial}{\partial w}\left[i\cos\theta\ln\left(\frac{1-iw}{1+iw}\right)-\ln w\right]=0, \quad \text{(B3)}$$

whence

$$w_{sp}=e^{\pm i\theta}, \quad \text{(B4)}$$

and the original contour γ_0 of integration enclosing the origin must be deformed to pass through the points $w=e^{\pm i\theta}$ along the directions of steepest descent and to avoid intersections with the branch cuts.

The contributions of both saddle points to the leading term of the asymptotic expansion of the integral (B2) are of the same order. Therefore

$$P_n^{(\lambda)}(x)=\frac{1}{2\pi i}\sum_{w_{sp}}f(w_{sp})\sqrt{-\frac{2\pi}{(n+1)S''(w_{sp})}}\times\exp\left[(n+1)S(w_{sp})\right], \quad \text{(B5)}$$

where

$$S(w_{sp})=\frac{\pi}{2}\left|\cos\theta\right|\mp i\left(\cos\theta\ln\left|\tan\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\right|+\theta\right), \quad \text{(B6)}$$

$$S''(w_{sp})=e^{\mp i\left(2\theta+\frac{\pi}{2}\right)}\tan\theta, \quad \text{(B7)}$$

and

$$f(w_{sp})=\frac{1}{\left(1+w_{sp}^2\right)^{\lambda}}=\frac{e^{\mp i\theta\lambda}}{(2\cos\theta)^{\lambda}}. \quad \text{(B8)}$$

Making use of Eqs. (B5)–(B8) we obtain the following asymptotic formula:

$$P_n^{(\lambda)}(x)\approx\frac{(n+1)^{\lambda-1}\exp(\pi|x|/2)}{(2|x|)^{\lambda-\frac{1}{2}}\sqrt{\pi}\left\{1-[x/(n+1)]^2\right\}^{1/4}}\times\sin\left[\frac{\pi}{4}+(n+\lambda)\arccos\left(\frac{x}{n+1}\right)+\frac{x}{2}\ln\left(\frac{1-\sqrt{1-[x/(n+1)]^2}}{1+\sqrt{1-[x/(n+1)]^2}}\right)\right]. \quad \text{(B9)}$$

2. Vicinity of the origin: $|x| \ll \sqrt{2n}$

At the origin $x \sim 0$ the saddle-point approximation used in the preceding section to calculate asymptotic value of the integral (B1) does not work, and the Plancherel-Rotach type formula fails. In this interval the Darboux method [17] turns out to be fruitful.

To obtain some asymptotic expansion for a polynomial P_n in accordance with the Darboux theorem we have to expand the corresponding generating function $F(x,w)$ in the vicinities of its singularities $e^{i\phi_k}$ on the unit circle $|w| = 1$ into series of the form

$$F(x,w) = \sum_{m=0}^{\infty} c_m^{(k)} \left(1 - we^{-i\phi_k}\right)^{a_k + mb_k}. \tag{B10}$$

Then the expression

$$\sum_{m=0}^{\infty} \sum_{k} c_m^{(k)} \binom{a_k + mb_k}{n} \left(-e^{i\phi_k}\right)^n \tag{B11}$$

furnishes an asymptotic expansion for $P_n(x)$.

For Pollaczek polynomials the singularities of the generating function Eq. (A7) occur at the points $w = \pm i$, in whose vicinities generating function can be expanded as

$$F(x,w) = 2^{-\lambda \pm ix} \sum_{m=0}^{\infty} \binom{-\lambda \pm ix}{m} \left(-\frac{1}{2}\right)^m \times (1 \pm iw)^{m-\lambda \pm ix}. \tag{B12}$$

Then the expression

$$(-1)^n \,\mathrm{Re}\, 2^{ix-\lambda+1} \sum_{m=0}^{\infty} \binom{ix-\lambda}{m} \binom{m-\lambda-ix}{n} \frac{e^{i\frac{\pi}{2}(2m-n)}}{2^m} \tag{B13}$$

leads to the asymptotic formula for Pollaczek polynomials.

The leading term results from $m = 0$ in Eq. (B13), so that for finite λ and $n \gg 1$

$$P_n^{(\lambda)}(x) \approx \left(\frac{n}{2}\right)^{\lambda-1} \frac{1}{|\Gamma(\lambda + ix)|} \times \cos\left(x \ln(2n) - \arg\Gamma(\lambda + ix) - \frac{\pi}{2}n\right). \tag{B14}$$

The next term $(m = 1)$ of asymptotic expansion Eq. (B13) is of the order

$$\mathcal{O}\{\sqrt{[x^2 + \lambda^2](x^2 + [\lambda - 1)^2]/2n}\}.$$

This circumstance imposes the restrictions $|\lambda| \ll \sqrt{2n}$ and $|x| \ll \sqrt{2n}$ for Eq. (B14).

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